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CAPACITATED FACILITY LOCATION: VALID INEQUALITIES AND FACETS

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Capacitated Facility Location: Valid Inequalities and Facets

by

Karen AARDAL*, Yves POCHET**
and Laurence A. WOLSEY**

June 1993, Revised February 1994

Abstract

We examine the polyhedral structure of the convex hull of feasible solutions of the capacitated facility location problem. In particular we derive necessary and sufficient conditions for a family of "effective capacity" inequalities to be facet-defining, and further results on a more general family called "submodular" inequalities.

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1. Introduction

The capacitated facility location (CFL) problem is a well-known combinatorial optimization problem. Here we examine the polyhedral structure of the convex hull of feasible solutions with a view to obtaining strong valid inequalities for use in a branch and bound or branch and cut algorithm. So far there has been relatively little work on the polyhedral structure of (CFL), apart from a paper of Leung and Magnanti (1989) for the case of constant capacities, a paper of Cornuéjols, Sridharan and Thizy (1991) comparing the strength of various relaxations, and a paper of Deng and Simchi-Levi (1993) examining the polyhedral structure of a related model. In contrast, the uncapacitated facility location (UFL) problem has been studied by Cornuéjols, Fisher and Nemhauser (1977), Guignard (1980), Cornuéjols and Thizy (1982), and Cho et al. (1983 a,b).

The contents of the paper are as follows. First we give the formulation of CFL, introduce necessary notation and general assumptions. In Section 2, we consider briefly inequalities known to be facet defining for two relaxations of CFL, namely the surrogate knapsack and single node flow polytopes. In Sections 3 and 4 we introduce two new families of inequalities; the family of Effective Capacity (EC) inequalities which can be viewed as a generalization of the well-known flow cover inequalities, and the family of submodular inequalities which in turn generalizes the EC inequalities. We give necessary and sufficient conditions for the EC inequalities and for some more general submodular structures to be facet defining. Finally, in Section 5 we discuss two more families of inequalities; the class of combinatorial inequalities introduced by Cho et al. (1983 a) for UFL, and the class of (k, l, S, I) inequalities developed for the lot-sizing problem with constant batch sizes by Pochet and Wolsey (1993). For the combinatorial inequalities we give sufficient conditions for them to be facet defining for CFL.

Let $M = \{1, ..., m\}$ be the set of facilities (depots) and $N = \{1, ..., n\}$ the set of clients. $y_j = 1$ if depot j is open, and $y_j = 0$ otherwise. For every $j \in M$ and $k \in N$, an arc (j, k) exists and v_{jk} denotes the flow from depot j to client k. Depot j has capacity m_j, and the demand of client k is d_k. The demand of the clients in the set S is denoted by $d(S)$. The fixed costs of opening depot j is f_j, and the cost of sending one unit of flow from j to k is c_{jk}. The objective is to minimize the sum of fixed costs and transportation costs. Below we give the standard formulation of CFL as a mixed integer programming problem:

$$z = \min\{\sum_{j \in M}\sum_{k \in N} c_{jk}v_{jk} + \sum_{j \in M} f_j y_j : (v, y) \in X^{CFL}\}$$

where X^{CFL} is defined by constraints (1.1)-(1.6) given below.

$$\sum_{j \in M} v_{jk} = d_k \qquad k \in N \tag{1.1}$$

$$\sum_{k \in N} v_{jk} \leq m_j y_j \qquad j \in M \tag{1.2}$$

$$v_{jk} \leq d_k y_j \qquad j \in M,\ k \in N \tag{1.3}$$

$$v_{jk} \geq 0 \qquad j \in M,\ k \in N \tag{1.4}$$

$$y_j \leq 1 \qquad j \in M \tag{1.5}$$

$$y_j \text{ integer} \qquad j \in M. \tag{1.6}$$

For modelling and computational purposes it is useful to introduce additional variables v_j representing the total flow leaving depot j with defining constraints $v_j = \sum_{k \in N} v_{jk}$, (1.7), to add the aggregate (and redundant) constraint

$$\sum_{j \in M} v_j = d(N), \tag{1.8}$$

and to replace (1.2) by the equivalent constraint

$$v_j \leq m_j y_j. \tag{1.9}$$

We assume throughout the paper that

$$\sum_{j \in M} m_j - m_r \geq d(N) \qquad \text{for all } r \in M. \tag{A1}$$

This assumption ensures that there exists a feasible solution with any single depot closed, and it is part of the hypotheses of all propositions that concern facets and the dimension of $conv(X^{CFL})$ established below. If all depots need to be open in every feasible solution, the problem reduces to the transportation problem.

PROPOSITION 1. $dim(conv(X^{CFL})) = m \times n + m - n$.

2. Knapsack and Flow Cover Facets

Here we examine two simple but important relaxations of CFL. Combining (1.8) and (1.9) with (1.5), (1.6) and $v_j \geq 0$, $y_j \geq 0$ for $j \in M$, we obtain

PROPOSITION 2. *The knapsack set*

$$X^K = \{y \in \{0,1\}^m : \sum_{j \in M} m_j y_j \geq d(N)\}$$

is a relaxation of X^{CFL}.

As valid inequalities for X^K are valid for X^{CFL}, and generating facet defining inequalities for X^K is "practically solved" (Crowder et al. (1983)), it is natural to examine whether facet defining inequalities for X^K are also facet defining for X^{CFL}. Let $J \subset M$ be a subset of depots such that $\sum_{j \in J} m_j > \sum_{j \in M} m_j - d(N)$, i.e. if all depots in J are closed then the demand cannot be met. J is called a *cover* with respect to M and N, and J is a *minimal cover* if in addition for all $S \subset J$, $\sum_{j \in S} m_j \leq \sum_{j \in M} m_j - d(N)$.

THEOREM 3. *If J is minimal cover with respect to M and N, $m_{\min} = \min_{j \in J}(m_j)$, and $\sum_{j \in M \setminus J} m_j + m_{\min} > d(N)$, then the knapsack cover inequality:*

$$\sum_{j \in J} y_j \geq 1 \tag{2.1}$$

defines a facet of $conv(X^{CFL}) \cap \{y \in \{0,1\}^m : y_j = 1 \text{ for } j \in M \setminus J\}$.

Proof. See Aardal (1992), Proposition 3.3.

A general family of facets for X^{CFL} is obtained by choosing a subset $M' \subset M$ of depots (i.e. initially $y_j = 0$ for $j \in M \setminus M'$), deriving a facet-defining inequality (2.1) from J' where J' is a minimal cover with respect to M' and N, and then applying standard sequential lifting procedures by lifting in the variables $y_j = 0$ for $j \in M \backslash M'$, and the variables $y_j = 1$ for $j \in M' \backslash J'$. The resulting facet-defining inequality for X^{CFL} is of the form:

$$\sum_{j \in J'} y_j \geq 1 - \sum_{j \in M \backslash M'} \alpha_j y_j + \sum_{j \in M' \backslash J'} \beta_j (1 - y_j)$$

for appropriately chosen values of $\alpha_j,\ \beta_j \geq 0$.

We now consider a second relaxation consisting of (1.8), (1.9), (1.5), (1.6) and $v_j \geq 0,\ y_j \geq 0$ for $j \in M$.

PROPOSITION 4. *The flow-cover set*

$$X^{FC} = \{(v, y) \in \mathbb{R}_+^m \times \mathbb{Z}_+^m : \sum_{j \in M} v_j = d(N), v_j \leq m_j y_j,\ y_j \leq 1,\ j \in M\}$$

is a relaxation of X^{CFL}.

As shown in greater generalization below, we obtain

THEOREM 5. *If* $J \subset M$ *is a flow cover with respect to* M *and* N, *i.e.* $\sum_{j \in J} m_j = d(N) + \lambda,\ \lambda > 0$, *with*

i) $\max_{j \in J}(m_j) > \lambda$,

ii) $\sum_{j \in M} m_j > d(N) + m_r$ *for all* $r \in J$,

then the flow cover inequality

$$\sum_{j \in J} v_j + \sum_{j \in J} (m_j - \lambda)^+ (1 - y_j) \leq d(N) \tag{2.2}$$

defines a facet of $conv(X^{CFL})$.

The specific interest of Theorems 3 and 5 is that the separation heuristics developed for knapsack cover and flow cover inequalities, see Van Roy and Wolsey (1987), are incorporated in some existing MPS systems such as MPSARX (Van Roy and Wolsey) and MINTO (Savelsbergh et al. (1991)), and can be applied directly to formulation

(1.1)-(1.9). We also observe that when the capacities are constant, $m_j = m$ for all $j \in M$, X^{FC} takes the form:

$$X_C^{FC} = \{(v,y) \in \mathbb{R}_+^m \times \mathbb{Z}_+^m : \sum_{j \in M} v_j = d(N),\ v_j \leq m y_j,\ y_j \leq 1,\ j \in M\}.$$

Assuming $d(N)$ is not an integer multiple of m, let $l = \lceil d(N)/m \rceil$ be the size of a flow cover satisfying Condition i) of Theorem 5, and $\lambda = ml - d(N) > 0$. The flow cover inequalities can in this constant capacity case be written as

$$\sum_{j \in S} v_j - \sum_{j \in S} (m - \lambda) y_j \leq d(N) - (m - \lambda) l \tag{2.3}$$

where S is any flow cover (i.e. $|S| \geq l$). Padberg et al. (1985) have given an explicit description of $conv(X_C^{FC})$ consisting of the initial constraints and an exponential number of facets of the form (2.3). Letting $\pi_j = \max\{0, v_j - (m - \lambda) y_j\}$, it is readily seen that an alternative is to use the following extended formulation.

$$\begin{aligned} Q = \{(\pi, v, y) \in \mathbb{R}_+^m \times \mathbb{R}_+^m \times \mathbb{Z}_+^m : & \sum_{j \in M} \pi_j \leq d(N) - (m - \lambda) l & \\ & \pi_j \geq v_j - (m - \lambda) y_j & j \in M \\ & \sum_{j \in M} v_j = d(N) & \\ & v_j \leq m_j y_j & j \in M \\ & y_j \leq 1 & j \in M\}. \end{aligned}$$

THEOREM 6. $proj_{v,y}(Q) = conv(X_C^{FC})$.

Proof. Eliminating the variables π_j gives the inequalities (2.3) for all $S \subseteq M$ plus the initial constraints.

■

3. Effective Capacity Inequalities

Here we first generalize the flow cover inequalities by choosing a subset $K \subseteq N$ of clients, a subset $J \subseteq M$ of depots, and a subset $K_j \subseteq K$ for each $j \in J$. Thus we are interested in equalities containing flows in the arc set $\{(j,k) : j \in J,\ k \in K_j\}$. When $K_j = K$ for all $j \in J$, and J is a flow cover with respect to M and K, the flow cover inequality

$$\sum_{j\in J}\Big(\sum_{k\in K} v_{jk}\Big) + \sum_{j\in J}(m_j - \lambda)^+(1 - y_j) \leq d(K) \tag{3.1}$$

is such an inequality.

By choosing a subset of arcs between J and K instead of the complete arc set, we are able to use the "effective capacity" $\overline{m}_j = \min(m_j, d(K_j))$. Thus, if $d(K_j) < m_j$ for at least one depot $j \in J$, it is possible to obtain a tighter inequality.

Given a subset of clients $K \subseteq N$, choose for each $j \in M$ a set $K_j \subseteq K$. We now say that $J \subset M$ is a *flow cover* if $\sum_{j\in J}\overline{m}_j = d(K) + \lambda,\ \lambda > 0$.

PROPOSITION 7. *Let $J \subset M$ be a flow cover with respect to M and K, and assume that $\max_{j\in J}(\overline{m}_j) > \lambda$. The Effective Capacity inequality*

$$\sum_{j\in J}\sum_{k\in K_j} v_{jk} + \sum_{j\in J}(\overline{m}_j - \lambda)^+(1 - y_j) \leq d(K) \tag{3.2}$$

is valid for X^{CFL}.

EXAMPLE 1.

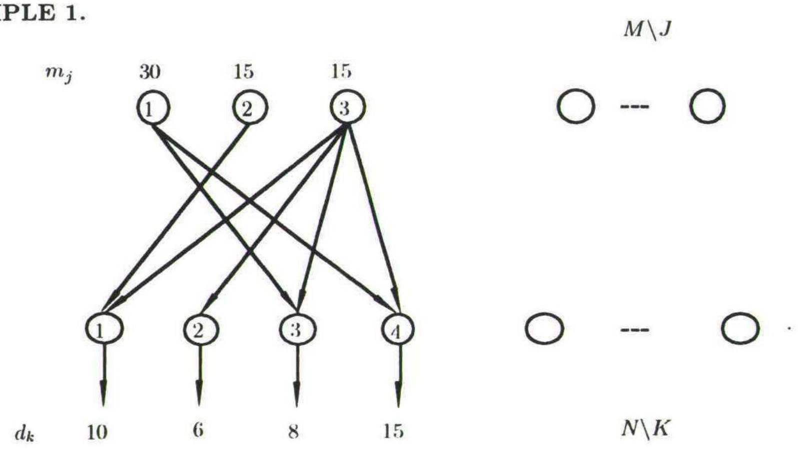

Figure 1.

Let $K = \{1,2,3,4\}$, $J = \{1,2,3\}$, $K_1 = \{3,4\}$, $K_2 = \{1\}$ and $K_3 = K$. The set J is a flow cover with respect to M and K, and the excess $\lambda = 9$. The Effective Capacity inequality

$$v_{13} + v_{14} + v_{21} + v_{31} + v_{32} + v_{33} + v_{34} + 14(1 - y_1) + (1 - y_2) + 6(1 - y_3) \leq 39$$

defines a facet of the convex hull of feasible solutions.

∎

The following result tells us under precisely what conditions these inequalities are facet defining.

THEOREM 8. *Let $J \subset M$ be a flow cover with respect to M and K, and let $Q \subset J$ be the subset of depots for which $\overline{m}_q < m_q$. Assume that $\sum_{j \in M} m_j > d(N) + m_r$ for all $r \in J$. The Effective Capacity inequality*

$$\sum_{j \in J} \sum_{k \in K_j} v_{jk} + \sum_{j \in J} (\overline{m}_j - \lambda)^+ (1 - y_j) \leq d(K)$$

definès a facet of $conv(X^{CFL})$ if and only if

- **a)** *for each pair of depots $q_1, q_2 \in Q$, $K_{q_1} \cap K_{q_2} = \emptyset$,*
- **b)** *$K_j = K$ for all $j \in J \backslash Q$,*
- **c)** *$(\cup_{q \in Q} K_q) \subset K$,*
- **d)** *$\overline{m}_q > \lambda$ for all $q \in Q$,*
- **e)** *if $|Q| \leq 1$, then $\exists j \in J \backslash Q$ with $\overline{m}_j = m_j > \lambda$.*

Proof.

Sufficiency:

This proof uses a standard technique, see for instance Nemhauser and Wolsey (1988), Section I.4.3., Theorem 3.6.

We show that the inequality

$$\sum_{j \in J} \sum_{k \in K_j} v_{jk} + \sum_{j \in J} (\overline{m}_j - \lambda)^+ (1 - y_j) \leq d(K) \tag{3.2}$$

plus any linear combination of the constraints $\sum_{j \in M} v_{jk} = d_k,\ k \in N$ is the only inequality that is satisfied with equality by all points $(v, y) \in X^{CFL}$ that are tight for (3.2), i.e. we show that if all tight points of X^{CFL} for (3.2) satisfy

$$\sum_{j \in M} \sum_{k \in N} \alpha_{jk} v_{jk} + \sum_{j \in M} \beta_j y_j = \alpha_0 \tag{$*$}$$

then

$$\begin{aligned}
&1)\ \beta_j = 0 && j \in M \setminus J \\
&2)\ \alpha_{jk} = \gamma_k && j \in M \setminus J,\ k \in K \\
&3)\ \alpha_{jk} = \gamma_k && j \in M,\ k \in N \setminus K \\
&4)\ \alpha_{jk} = \gamma_k && j \in J,\ k \in K \setminus K_j \\
&5)\ \alpha_{jk} = \gamma_k + \overline{\alpha} && j \in J,\ k \in K_j \\
&6)\ \beta_j = -\overline{\alpha}(\overline{m}_j - \lambda)^+ && j \in J \\
&7)\ \alpha_0 = \overline{\alpha}\Big(d(K) - \sum_{j \in J} (\overline{m}_j - \lambda)^+\Big) + \sum_{k \in N} \gamma_k d_k.
\end{aligned}$$

In the proof we consider three different types of tight points. These points are solutions $(v, y) \in X^{CFL}$ that are subject to the additional systems of contraints given below. Let $\varepsilon > 0$, and recall that Q is the set of depots for which $\overline{m}_q < m_q,\ K_q \subset K$.

(i) **All depots in M are open.**

$$\begin{aligned}
\sum_{j \in J} \sum_{k \in K_j} v_{jk} &= d(K) && \\
\sum_{j \in M} \sum_{k \in N \setminus K} v_{jk} &= d(N \setminus K) && \\
\sum_{k \in N} v_{jk} &\leq (m_j - \varepsilon) && j \in M \\
y_j &= 1 && j \in M \\
v_{jk} &\geq \varepsilon && j \in M,\ k \in N \setminus K \\
& && j \in J,\ k \in K_j.
\end{aligned}$$

(ii) **One depot $j_1 \in M \backslash J$ is closed.**

$$
\begin{aligned}
\sum_{j\in J}\sum_{k\in K_j} v_{jk} &= d(K) \\
\sum_{j\in M\backslash\{j_1\}}\sum_{k\in N\backslash K} v_{jk} &= d(N\backslash K) \\
y_j &= 1 && j \in M\backslash\{j_1\} \\
y_{j_1} &= 0.
\end{aligned}
$$

(iii) **One depot $j_1 \in J$ with $(\overline{m}_{j_1} - \lambda) > 0$ is closed.**

$$
\begin{aligned}
\sum_{k\in K_j} v_{jk} &= \overline{m}_j = d(K_j) && j \in Q\backslash\{j_1\} \\
\sum_{j\in(J\backslash Q)\backslash\{j_1\}}\ \sum_{k\in K\backslash\cup_{j\in Q\backslash\{j_1\}}K_j} v_{jk} &= d(K) - (\overline{m}_{j_1} - \lambda)^+ - \sum_{j\in Q\backslash\{j_1\}} d(K_j) \\
\sum_{j\in M\backslash J}\ \sum_{k\in K\backslash\cup_{j\in Q\backslash\{j_1\}}K_j} v_{jk} + \sum_{j\in Q\backslash\{j_1\}}\ \sum_{k\in K\backslash\cup_{j\in Q\backslash\{j_1\}}K_j} v_{jk} &= (\overline{m}_{j_1} - \lambda)^+ \\
\sum_{j\in(M\backslash J)\cup(Q\backslash\{j_1\})}\ \sum_{k\in N\backslash K} v_{jk} &= d(N\backslash K) \\
\sum_{k\in N} v_{jk} &\le m_j - \varepsilon && j \in M\backslash J \\
y_j &= 1 && j \in M\backslash\{j_1\} \\
y_{j_1} &= 0 \\
v_{jk} &\ge \varepsilon \\
&j \in (J\backslash Q)\backslash\{j_1\}, k \in K\backslash \cup_{j\in Q\backslash\{j_1\}} K_j \\
&j \in M\backslash J,\ k \in N\backslash \cup_{j\in Q\backslash\{j_1\}} K_j \\
&j \in Q\backslash\{j_1\},\ k \in (N\backslash \cup_{l\in Q\backslash\{j\}} K_l) \cup K_{j_1}.
\end{aligned}
$$

A feasible solution to systems (i) and (ii) exists due to Assumption (A1), and a feasible solution to system (iii) exists due to the assumption that $\sum_{j\in M} m_j > d(N) + m_r$ for all $r \in J$, and due to the conditions given in the theorem. The structure of a solution to system (iii) in the case that $Q \neq \emptyset$ is shown in Figure 2.

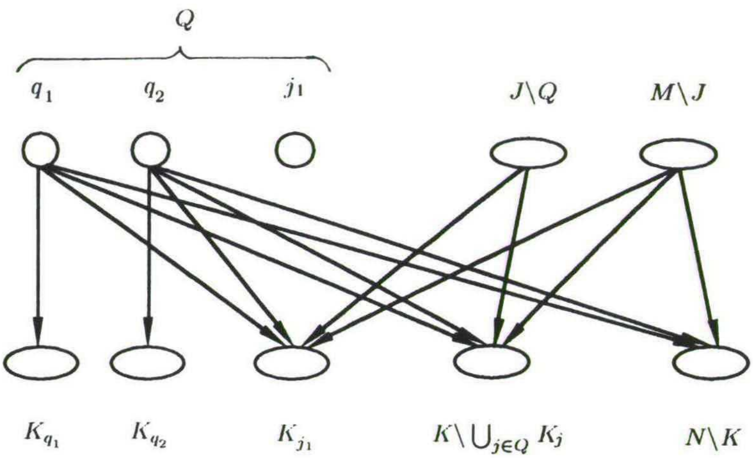

Figure 2.

The general idea of the proof technique is as follows. In order to establish the values of the coefficients, α_{jk}, β_j, and α_0 according to points 1)–7) above, we construct a feasible solution to an appropriate system of constraints (i), (ii) or (iii). Then, a small change in the solution is made. By evaluating $(*)$ at both solutions and by comparing the resulting expressions, the possible values of a set of coefficients are obtained.

We start by showing that

1) $\quad \beta_j = 0, \quad j \in M\backslash J.$

Consider any feasible solution to system (ii) where j_1 is any depot in the set $M\backslash J$. Take the same solutions but with $y_{j_1} = 1$. This gives $\beta_{j_1} = 0$. By varying over all possible choices of j_1 we obtain

$$\beta_j = 0, \quad j \in M\backslash J.$$

Next, show that

2) $\quad \alpha_{jk} = \alpha_k^1, \; j \in M\backslash J, \; k \in K.$

Consider a solution to constraint system (iii) with the choice of depot j_1 given below. Recall that from the definition of λ, any solution to this system satisfies $\sum_{k\in K_j} v_{jk} = \overline{m}_j, \; j \in J\backslash\{j_1\}$. Given that $(\overline{m}_{j_1} - \lambda)^+ > 0$, $\sum_{j\in J\backslash\{j_1\}} \sum_{k\in K_j} v_{jk} =$

$\sum_{j\in J\setminus\{j_1\}} \overline{m}_j < d(K)$. Since the clients in K are not saturated by flow from depots in J, it is possible to have flow from the set $M\setminus J$ to the client set K.

Case 1: $Q = \emptyset$. Therefore $\overline{m}_j = m_j$ for all $j \in J$. From Condition b) $K_j = K$ for all $j \in J$ and from Condition e), there exists at least one depot in J with $(m_j - \lambda) > 0$. Let depot j_1 be any depot such that $(m_{j_1} - \lambda) > 0$.

Case 2: $Q = \{q\}$. Thus $K_q \subset K$, and by Condition d), $\overline{m}_q > \lambda$. Let $j_1 = q$.

Case 3: There exists a subset of depots $Q \subset J$, $|Q| > 1$ with $\overline{m}_q < m_q$. Therefore, $K_q \subset K$ for all $q \in Q$ due to Condition c), and for any pair of depots $q_1, q_2 \in Q$, $K_{q_1} \cap K_{q_2} = \emptyset$ due to Condition a). Moreover, from d), $\overline{m}_q > \lambda$ for all $q \in Q$. Let j_1 be any depot in Q.

Take any two depots $j', j'' \in M\setminus J$ and any client $k' \in K\setminus \bigcup_{j\in Q\setminus\{j_1\}} K_j$. Make an ε-change of flow between the depots and the client, and repeat for all possible combinations of depots and clients and, if $|Q| > 1$, for all possible choices of depot j_1. This gives

$$\alpha_{jk} = \alpha_k^1, \ j \in M\setminus J, \ k \in K.$$

Next, show that

3) $\qquad \alpha_{jk} = \alpha_k^2, \ j \in M, \ k \in N\setminus K.$

Consider any solution to constraint system (i). Choose any client in $N\setminus K$ and any two depots in M. Make an ε-change of flow between the two depots and the client, and repeat for all possible choices of depots and clients. This gives

$$\alpha_{jk} = \alpha_k^2, \ j \in M, \ k \in N\setminus K.$$

4) $\qquad$ Show that for any $j \in Q$, $\alpha_{jk} = \alpha_k^1$ for $k \in K\setminus K_j$.

Consider a solution to constraint system (iii).

Case 1: $Q = \{q\}$. Thus $K_q \subset K$ due to Condition c). Due to Condition e) there exists at least one depot $r \in J\setminus\{q\}$ with $\overline{m}_r > \lambda$. Let the depot j_1 be any depot in $J\setminus\{q\}$ having $\overline{m}_j > \lambda$. Choose any depot $j' \in M\setminus J$ and any client $k' \in K\setminus K_q$. Make an ε-change flow on the arcs (j', k') and (q, k'). This gives $\alpha_{qk'} = \alpha_{k'}^1$. Repeating for all possible $k' \in K\setminus K_q$ we get

$$\alpha_{qk} = \alpha_k^1, \ k \in K\setminus K_q.$$

Case 2: There exists a subset of depots $Q \subset J$, $|Q| > 1$. Hence, $K_q \subset K$, $q \in Q$ due to Condition c), and by Condition d), $\overline{m}_q > \lambda$, $q \in Q$. Let j_1 be any depot in the set Q. Choose two depots $j^{'} \in Q\backslash\{j_1\}$ and $j^{''} \in M\backslash J$ and a client $k^{'} \in K_{j_1} \cup (K\backslash \cup_{j\in Q} K_j)$. Make an ε-change of flow between the two depots $j^{'}, j^{''}$ and the client. Varying over possible choices of depots and clients gives $\alpha_{jk} = \alpha_k^1$, for $j \in Q\backslash\{j_1\}$, $k \in K_{j_1} \cup (K\backslash \cup_{j\in Q} K_j)$. Moreover j_1 can be chosen as any depot in Q, and hence

$$\alpha_{jk} = \alpha_k^1, \quad j \in Q,\ k \in K\backslash K_j.$$

5) Show that $\alpha_{jk} = \alpha_k^1 + \overline{\alpha}$, $j \in J$, $k \in K_j$.

Each client in $\cup_{j\in Q} K_j$ belongs to one set K_j and is also served by every depot in $J\backslash Q \neq \emptyset$. Therefore, the only possibility of having a client served by only one depot is if this client belongs to the set $(K\backslash \cup_{j\in Q} K_j) \neq \emptyset$ and if $|J\backslash Q| = 1$, in which case there is nothing to show for this specific client.

Consider any solution to constraint system (i). For any client $k^{'} \in K$ served by at least two depots, we can choose any two depots $j^{'}, j^{''} \in J$ such that $K_{j^{'}}, K_{j^{''}} \ni k^{'}$. Make an ε-change of flow between the depots and the client, and repeat for all possible choices of clients and depots for which $K_j \ni k$. This shows that

$$\alpha_{jk} = \alpha_k^3, \quad j \in J,\ k \in K_j.$$

Let $\alpha_k^3 = \alpha_k^1 + \overline{\alpha}_k$. Next we show that $\overline{\alpha}_k = \overline{\alpha}$. Consider solutions to constraint system (iii) with the choices of j_1 given below.

Case 1: $Q = \emptyset$. Let depot j_1 be any depot in J having $m_j > \lambda$. At least one such depot exists due to Condition e).

Case 2: $Q \neq \emptyset$. Let j_1 be any depot in Q. Due to Condition d) $\overline{m}_q > \lambda$ for all $q \in Q$.

Choose any two clients $k^{'}, k^{''}$ such that $k^{'} \in K_{j_1}$ and $k^{''} \in \cup(K\backslash \cup_{j\in Q} K_j)$, and any depots $j^{'}, j^{''}$ such that $j^{'} \in J\backslash(Q \cup \{j_1\})$ and $j^{''} \in M\backslash J$. Note that if $Q = \emptyset$, then $K_{j_1} = K$, and the clients $k^{'}$ and $k^{''}$ can be chosen arbitrarily among the clients in K. Due to Condition c) $\cup_{j\in Q} K_j \subset K$, i.e. $K\backslash \cup_{j\in Q} K_j \neq \emptyset$, so we know that the set of clients $K_{j_1} \cup (K\backslash \cup_{j\in Q} K_j)$ consists of at least two clients.

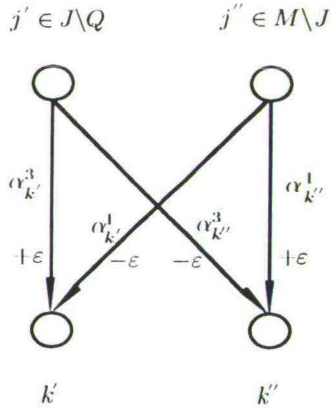

Figure 3.

Increase the flow on the arcs $(j^{'}, k^{'})$ and $(j^{''}, k^{''})$ by ε, and decrease the flow on arcs $(j^{'}, k^{''})$ and $(j^{''}, k^{'})$ by ε. This gives

$$\alpha^3_{k'} - \alpha^1_{k'} - \alpha^3_{k''} + \alpha^1_{k''} = 0 \tag{1}$$

By again using $\alpha^3_{k'} = \overline{\alpha}_{k'} + \alpha^1_{k'}$ and $\alpha^3_{k''} = \overline{\alpha}_{k''} + \alpha^1_{k''}$, we obtain

$$\overline{\alpha}_{k'} \quad \overline{\alpha}_{k''}.$$

If $Q \neq \emptyset$, as $K \backslash \cup_{j \in Q} K_j \neq \emptyset$, the sets $K_{j_1} \cup (K \backslash \cup_{j \in Q} K_j)$ for j_1 being in turn each depot in Q, always have at least one element in common. Since $k^{'}, k^{''}$ can be chosen as any clients in $K_{j_1} \cup (K \backslash \cup_{j \in Q} K_j)$ we can conclude that

$$\overline{\alpha}_k = \overline{\alpha}, \quad k \in K.$$

For simplicity of notation we now define $\gamma_k = \alpha^1_k$ for $k \in K$ and $\gamma_k = \alpha^2_k$ for $k \in N \setminus K$.

6) Show that $\beta_j = -\overline{\alpha}(\overline{m}_j - \lambda)^+$, $j \in J$.

The hyperplane $(*)$ given in the beginning of the proof has now been reduced to

$$\overline{\alpha} \sum_{j \in J} \sum_{k \in K_j} v_{jk} + \sum_{k \in N} \gamma_k \sum_{j \in M} v_{jk} + \sum_{j \in J} \beta_j y_j = \alpha_0. \tag{**}$$

Evaluate $(**)$ at any solution to system (i) and any tight feasible solution in which one depot in J is closed, and take the difference between $(**)$ evaluated at these two solutions. This gives $\overline{\alpha}(-(\overline{m}_{j_1}-\lambda)^+)-\beta_{j_1}=0$, and since j_1 is any depot in J we have

$$\beta_j=-\overline{\alpha}(\overline{m}_j-\lambda)^+,\ j\in J.$$

7) Determine the value of α_0.

By using the value of β_j, $(**)$ can be rewritten as $(**)'$

$$\overline{\alpha}(\sum_{j\in J}\sum_{k\in K_j}v_{jk}-\sum_{j\in J}(\overline{m}_j-\lambda)^+y_j)+\sum_{k\in N}\gamma_k\sum_{j\in M}v_{jk}=\alpha_0. \qquad (**)'$$

Evaluating $(**)'$ at any point $(v,y)\in X_{SFL}$ that is tight for (3.2) gives

$$\overline{\alpha}(d(K)-\sum_{j\in J}(\overline{m}_j-\lambda)^+)+\sum_{k\in N}\gamma_k d_k=\alpha_0,$$

and we have completed the proof that (3.2) defines a facet for $conv(X^{CFL})$ under Conditions a)-e).

Necessity:

Let $J^+=\{j\in J:\overline{m}_j>\lambda\}$. If $J^+=\emptyset$, the inequality (3.2) becomes

$$\sum_{j\in J}\sum_{k\in K_j}v_{jk}\le d(K)$$

which is dominated by a combination of constraints in the problem formulation. Hence, we can assume that $J^+\neq\emptyset$.

The tight points of inequality (3.2)

$$\sum_{j\in J}\sum_{k\in K_j}v_{jk}\le d(K)-\sum_{j\in J^+}(\overline{m}_j-\lambda)(1-y_j)$$

are feasible points such that

either $\quad i)$ $y_j=1,\ j\in J^+$ and $\sum_{j\in J}\sum_{k\in K_j}v_{jk}=d(K)$, in which case $v_{jk}=0$, $j\in M\backslash J,\ k\in K$,

or $\quad ii)$ $y_j=1,\ j\in J^+\backslash\{p\},\ y_p=0$ and $\sum_{j\in J\backslash\{p\}}\sum_{k\in K_j}v_{jk}=d(K)-(\overline{m}_p-\lambda)^+=\sum_{j\in J\backslash\{p\}}\overline{m}_j$, where the last equality follows from the definition of λ.

(1a) Assume that $K_{q_1} \cap K_{q_2} \neq \emptyset$.

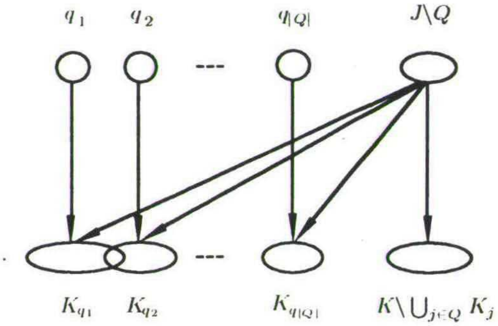

Figure 4.

Case 1: $J^+\backslash\{q_1, q_2\} \neq \emptyset$.

Choose $t \in J^+\backslash\{q_1, q_2\}$. For the tight points in (i) $y_t = 1$. For the tight points in (ii) either $p \neq t$, so $y_t = 1$ or $p = t$ in which case

$$\sum_{j\in J\backslash\{t\}} \sum_{k\in K_j} v_{jk} = \sum_{j\in J\backslash\{t\}} \overline{m}_j,$$

which implies that $\sum_{k\in K_j} v_{jk} = \overline{m}_j$ for all $j \in J\backslash\{t\}$. Since $\overline{m}_{q_i} = d(K_{q_i}),\ i = 1,2$ this requires $\sum_{k\in K_{q_i}} v_{q_i k} = d(K_{q_i}),\ i = 1,2$ which is impossible since $K_{q_1} \cap K_{q_2} \neq \emptyset$. Hence the tight points in (ii) all have $y_t = 1$. Therefore, all tight points have $y_t = 1$ which implies that (3.2) is not a facet in this case.

Case 2: $J^+\backslash\{q_1, q_2\} = \emptyset$.

For the tight points in (i) $v_{jk} = 0,\ j \in M\backslash J,\ k \in K$. In (ii), let p be any depot in J^+, i.e. $p \in J^+ \cap \{q_1, q_2\}$. For all $k \in K_{q_1} \cap K_{q_2}$, $v_{q_i k} = d_k$ for $q_i \in \{q_1, q_2\}\backslash\{p\}$ which implies $v_{jk} = 0,\ j \in M\backslash J,\ k \in K_{q_1} \cap K_{q_2}$. Hence, all tight points have $v_{jk} = 0,\ j \in M\backslash J,\ k \in K_{q_1} \cap K_{q_2}$, which means that (3.2) cannot define a facet in this case.

(1b) Assume that $K_l \subset K$ for some depot $l \in J\backslash Q$.

The inequality based on this structure

$$\sum_{j\in J\backslash\{l\}} \sum_{k\in K_j} v_{jk} + \sum_{k\in K_l} v_{lk} + \sum_{j\in J} (\overline{m}_j - \lambda)^+(1 - y_j) \leq d(K)$$

is dominated by the valid inequality

$$\sum_{j\in J\setminus\{l\}}\sum_{k\in K_j} v_{jk} + \sum_{k\in K} v_{lk} + \sum_{j\in J}(\overline{m}_j - \lambda)^+(1-y_j) \le d(K)$$

as $K_l \subset K$ and $\overline{m}_l = m_l$. Thus (3.2) cannot define a facet in this case.

(1c) Assume that $\bigcup_{j\in Q} K_j = K$, and that the sets $\{K_j\}_{j\in Q}$ are disjoint.

The value of $\lambda = \sum_{j\in J}\overline{m}_j - d(K) = \sum_{j\in J\setminus Q}\overline{m}_j$. Therefore $\overline{m}_j \le \lambda$ for all $j \in J\setminus Q$ and (3.2) will be of the following form.

$$\sum_{j\in J}\sum_{k\in K_j} v_{jk} + \sum_{j\in Q}(\overline{m}_j - \lambda)^+(1-y_j) \le d(\bigcup_{j\in Q} K_j). \tag{1}$$

Inequality (1) can be viewed as the sum over q in Q of the EC inequalities having node cover $\{q\}\cup J\setminus Q$, client set K_q and excess capacity $\overline{m}_q + \sum_{j\in J\setminus Q} m_j - d(K_q) = \lambda$ from above, namely

$$\sum_{k\in K_q} v_{q_t k} + \sum_{j\in J\setminus Q}\sum_{k\in K_q} v_{jk} + (\overline{m}_{q_t} - \lambda)^+(1-y_q) \le d(K_q).$$

Hence (3.2) cannot define a facet if $\bigcup_{j\in Q} K_j = K$.

(1d) Assume that $\overline{m}_q \le \lambda$ for some $q \in Q$.

For the tight points in (i) $v_{jk} = 0,\ j \in M\setminus J,\ k \in K$. For the tight points in (ii) take $p \in J\setminus\{q\}$. Then

$$\sum_{j\in J\setminus\{p\}}\sum_{k\in K_j} v_{jk} = \sum_{j\in J\setminus\{p\}}\overline{m}_j$$

which implies $\sum_{k\in K_q} v_{qk} = \overline{m}_q = d(K_q)$. Hence, $v_{jk} = 0,\ j \in M\setminus J,\ k \in K_q$. When $p = q$, $\sum_{j\in J\setminus\{q\}}\sum_{k\in K_j} v_{jk} = d(K)$ and again $v_{jk} = 0$ for $j \in M\setminus J,\ k \in K_q$. Thus, all tight points satisfy $v_{jk} = 0\ j \in M\setminus J,\ k \in K_q$ which implies that (3.2) does not define a facet if $\overline{m}_q \le \lambda$ for some $q \in Q$.

(1e) Assume that $Q = \{q\}$, i.e $|Q| = 1$, $m_j \le \lambda$ for all $j \in J\setminus Q$ and $\overline{m}_q > \lambda$.

Inequality (3.2) based on this structure is

$$\sum_{k\in K_q} v_{qk} + \sum_{j\in J\setminus\{q\}}\sum_{k\in K} v_{jk} + (\overline{m}_q - \lambda)(1-y_q) \le d(K).$$

Replacing K_q by K gives a new value λ' for the excess of $\lambda' = \lambda + (m_q - \overline{m}_q) > \lambda$. Therefore $(m_j - \lambda')^+ = 0,\ j \in J\backslash\{q\}$ while $m_q - \lambda' = m_q - \lambda - m_q + \overline{m}_q = \overline{m}_q - \lambda$. The resulting flow cover inequality

$$\sum_{j\in J}\sum_{k\in K} v_{jk} + (\overline{m}_q - \lambda)(1 - y_q) \leq d(K)$$

is stronger than the original inequality which shows that (3.2) cannot define a facet if $|Q| = 1$ and $m_j \leq \lambda$ for all $j \in J\backslash Q$.

■

4. Submodular Inequalities

The question now is whether the EC inequalities (3.2) are the strongest inequalities of the form

$$\sum_{j\in J}\sum_{k\in K_j} v_{jk} + \sum_{j\in J} \beta_j(1 - y_j) \leq d(K). \tag{4.1}$$

A partial answer is provided below. We describe a class of inequalities that in an important special case can be written in the same form as (4.1), with $\beta_j \geq (\overline{m}_j - \lambda)^+$ for all $j \in J$, and which contains facet defining inequalities with $\beta_j > (\overline{m}_j - \lambda)^+$ for at least one $j \in J$. This family, called the family of submodular inequalities, was introduced in a general form for fixed-charge network problems by Wolsey (1989).

DEFINITION 10. *A set function f on $N = \{1, ..., n\}$ is submodular if*

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

for all $A, B \subseteq N$.

Let $\rho_j(A) = f(A \cup \{j\}) - f(A)$ for $j \in N\backslash A$ be the increment function.

PROPOSITION 11. *(Welsh (1976)). f is submodular if and only if $\rho_j(A) \geq \rho_j(B)$ for all $A \subseteq B \subseteq N\backslash\{j\}$.*

PROPOSITION 12. *Let $K \subseteq N$, $J \subseteq M$ and $K_j \subseteq K$ for all $j \in J$. The function*

$$
\begin{aligned}
f(J) = \max\{\sum_{j \in J} \sum_{k \in K_j} v_{jk} : & & (4.2)\\
\sum_{k \in K_j} v_{jk} \leq \overline{m}_j y_j & \quad & j \in J\\
\sum_{\{j \in J : K_j \ni k\}} v_{jk} \leq d_k & & k \in K\\
v_{jk} \geq 0 & & j \in J,\ k \in K\\
y_j = 1 & & j \in J\}.
\end{aligned}
$$

is submodular on M.

The value of $f(J)$ is exactly the maximum flow from the depot set J to the client set K given the arc set $\{(j,k) : j \in J,\ k \in K_j\}$.

PROPOSITION 13. *Let $K \subseteq N$, $J \subseteq M$ and $K_j \subseteq K$ for all $j \in J$. The submodular inequality*

$$\sum_{j \in J} \sum_{k \in K_j} v_{jk} + \sum_{j \in J} \rho_j(J \backslash \{j\})(1 - y_j) \leq f(J) \tag{4.3}$$

is valid for $conv(X^{CFL})$.

For the submodular inequalities (4.3) to be valid, we do not require the set J to be a cover. However, $f(J) = d(K)$ is an important case and by making this assumption we can compare inequalities (4.3) to the EC inequalities (3.2).

PROPOSITION 14. *Given sets J, K and K_j, such that $f(J) = d(K)$, then the submodular inequality*

$$\sum_{j \in J} \sum_{k \in K_j} v_{jk} + \sum_{j \in J} \rho_j(J \backslash \{j\})(1 - y_j) \leq f(J) \tag{4.3}$$

is at least as strong as the EC inequality

$$\sum_{j \in J} \sum_{k \in K_j} v_{jk} + \sum_{j \in J} (\overline{m}_j - \lambda)^+(1 - y_j) \leq d(K). \tag{3.2}$$

Proof.

As $f(J) = d(K)$ by assumption, we only need to show that $\rho_j(J\backslash\{j\}) \geq (\overline{m}_j - \lambda)^+$. $\sum_{l \in J\backslash\{j\}} \overline{m}_l$ is an upper bound on the maximum flow $f(J\backslash\{j\})$ between the depots in $J\backslash\{j\}$ and the clients in K giving

$$\rho_j(J\backslash\{j\}) = f(J) - f(J\backslash\{j\}) = d(K) - f(J\backslash\{j\}) \geq (d(K) - \sum_{l \in J\backslash\{j\}} \overline{m}_l)^+ =$$

$$(\textstyle\sum_{l \in J} \overline{m}_l - \lambda - \sum_{l \in J\backslash\{j\}} \overline{m}_l)^+ = (\overline{m}_j - \lambda)^+.$$

∎

A consequence of Proposition 14 is that facet-defining EC-inequalities are also facet-defining submodular inequalities. Below we show that the class of facet-defining submodular inequalities *strictly* contains the class of facet-defining EC-inequalities by introducing two different structures which are generalizations of the facet-defining EC structure as defined in Theorem 8, and which both have $\rho_j(J\backslash\{j\}) > (\overline{m}_j - \lambda)^+$ for at least one $j \in J$. Moreover, for both structures it is possible to obtain a closed-form expression for the the maximum flow $f(J\backslash\{j\})$, and hence for $\rho_j(J\backslash\{j\})$ for all $j \in J$, which makes it possible to derive necessary and sufficient conditions for them to be facet-defining. Determining $f(J\backslash\{j\})$ for an arbitrary choice of sets J, K and K_j requires the use of a maximum flow algorithm. The motivation behind considering the two particular structures comes from the following observation.

OBSERVATION 15. *Consider a submodular inequality (4.3) for which $f(J) = d(K)$ and whose support graph is connected. If $\rho_j(J\backslash\{j\}) > (\overline{m}_j - \lambda)^+$ for some $j \in J$, then there exists a nontrivial partition of the clients $(K', K\backslash K')$ and the depots $(J', J\backslash J')$ with $j \in J'$ and with the clients in K' uniquely served by the depots in J'.*

Proof.

To calculate $\rho_j(J\backslash\{j\})$ we need to determine the value of $f(J\backslash\{j\})$, which is done by solving the maximum flow problem, or equivalently finding the minimum cut separating s and t in the graph below.

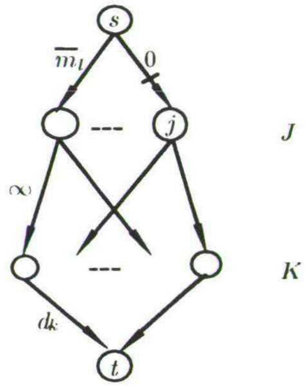

Figure 5.

The capacity on arcs (s,l), $l \in J\backslash\{j\}$ is $\overline{m}_l$ and arc (s,j) has capacity zero. On arcs (k,t), $k \in K$ the capacity is d_k, and all other arcs have infinite capacity. Let (L,R) be a minimum cut. Note that $j \in R$. The following cuts (L,R) are possible.

i) $L = \{s\}$, $R = J \cup K \cup \{t\}$,

ii) $L = \{s\} \cup (J\backslash\{j\}) \cup K$, $R = \{j,t\}$,

iii) $L = \{s\} \cup (J\backslash J^{'})$, $R = J^{'} \cup K \cup \{t\}$, $J^{'} \subset J$, $j \in J^{'}$,

iv) $L = \{s\} \cup K^{'}$, $R = J \cup (K\backslash K^{'}) \cup \{t\}$, $K^{'} \neq \emptyset$,

v) $L = \{s\} \cup (J\backslash J^{'}) \cup K$, $R = J^{'} \cup \{t\}$, $J^{'} \subset J$, $j \in J^{'}$,

vi) $L = \{s\} \cup (J\backslash J^{'}) \cup (K\backslash K^{'})$, $R = J^{'} \cup K^{'} \cup \{t\}$, $J^{'} \subset J$, $K^{'} \subset K$, $K^{'} \neq \emptyset$, $j \in J^{'}$.

The minimum cuts represented by *i)* and *ii)* give $\rho_j(J\backslash\{j\}) = (\overline{m}_j - \lambda)^+ > 0$ and $\rho_j(J\backslash\{j\}) = (\overline{m}_j - \lambda)^+ = 0$ respectively, i.e. $\rho_j(J\backslash\{j\}) = (\overline{m}_j - \lambda)^+$. Cut *iii)* has infinite capacity as all depots in $J\backslash J^{'}$ serve at least one client in K. Cut *iv)* cannot be minimum as the cut capacity $\sum_{l \in J\backslash\{j\}} \overline{m}_l + d(K^{'}) > \sum_{l \in J\backslash\{j\}} \overline{m}_l$ which is the capacity of cut *i)*. In case *v)* the cut capacity is equal to $\sum_{l \in J^{'}} \overline{m}_l + d(K)$ which is greater than the cut capacity $d(K)$ of cut *ii)* unless $J^{'} = \{j\}$ in which case we have cut *ii)*. Cut *vi)* has capacity less than ∞ only if there exists no arc (j,k) with $j \in L$ and $k \in R$. Hence, all clients in $K^{'}$ must be served by clients in $J^{'}$ only.

■

The first structure that we consider, having $\rho_j(J\backslash\{j\}) > (\overline{m}_j - \lambda)^+$ for at least one $j \in J$, is called the single-depot structure, and consists of a facet-defining EC-component, an additional depot set P and client sets $K_p^{'}$, $p \in P$ where the clients in $K_p^{'}$ are served by depot p only.

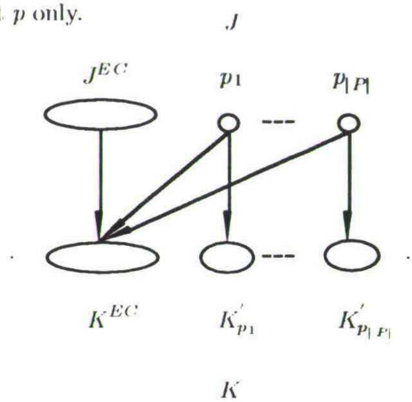

Figure 6.

For the single-depot structure the following holds.

LEMMA 16. *Let C^{EC} be an EC-component with client set K^{EC}, depot set J^{EC} and arc set $\{(j,k) : j \in J^{EC},\ k \in K_j \subseteq K^{EC}\}$ and such that J^{EC}, K^{EC} and $\{K_j\}_{j\in J^{EC}}$ satisfy the conditions of Theorem 8. The set $Q^{EC} \subset J^{EC}$ is the set of depots in J^{EC} having $\overline{m}_j < m_j$. Let P be a set of additional depots with client set $K_p = K^{EC} \cup K_p^{'}$, $K_p^{'} \neq \emptyset$, $p \in P$ where the clients in $K_p^{'}$ are served by depot p only and such that $m_p > d(K_p^{'})$ for all $p \in P$. Let $J = J^{EC} \cup P$, $K = K^{EC} \bigcup(\cup_{p\in P} K_p^{'})$ and $\lambda = \sum_{j\in J} \overline{m}_j - d(K)$.*

Then $\rho_j(J\backslash\{j\}) = (\overline{m}_j - \lambda)^+$ for all $j \in J^{EC}$ and $\rho_p(J\backslash\{p\}) = d(K_p^{'}) > (\overline{m}_p - \lambda)^+$ for all $p \in P$.

Proof.

Close depot r.

Case 1: $r \in P$.

If $r \in P$ we have $\sum_{l\in J\backslash\{r\}} \overline{m}_l > d(\cup_{l\in J\backslash\{r\}} K_l)$. Since $\overline{m}_q = d(K_q)$ for all $q \in Q^{EC}$, $\overline{m}_p > d(K_p^{'})$ for all $p \in P$, and since all clients in $K^{EC}\backslash(\cup_{j\in Q^{EC}} K_j^{EC})$ are served by all depots in $J^{EC}\backslash Q^{EC}$ and in $P\backslash\{r\}$, all demand of the clients in

$\{\cup_{l\in J\backslash\{r\}} K_l\}$ can be satisfied. However, no demand of the clients in $K_r^{'}$ can be satisfied as these clients are uniquely served by depot r. Thus,

$$f(J\backslash\{r\}) = d(\cup_{l\in J\backslash\{r\}} K_l) \text{ and } \rho_r(J\backslash\{r\}) = d(K_r^{'}).$$

$(\overline{m}_r - \lambda)^+ = (d(K) - \sum_{l\in J\backslash\{r\}} \overline{m}_l)^+ = (d(\cup_{l\in J\backslash\{r\}} K_l) + d(K_r^{'}) - \sum_{l\in J\backslash\{r\}} \overline{m}_l)^+ < d(K_r^{'}) = \rho_r(J\backslash\{r\})$

where the inequality follows from $\sum_{l\in J\backslash\{r\}} \overline{m}_l > d(\cup_{l\in J\backslash\{r\}} K_l)$.

Case 2: $r \in J^{EC}$ and $\{J^{EC}\backslash Q^{EC}\} = \{r\}$.

As C^{EC} is a facet-defining EC component we have $K_r^{'} = K_r \backslash \cup_{l\in(J\backslash\{r\})} K_l \neq \emptyset$ due to Condition c) of Theorem 8. Since $\overline{m}_q = d(K_q)$ for all $q \in Q^{EC}$ and $m_p > d(K_p^{'})$ for all $p \in P$, all clients in $\cup_{q\in Q^{EC}} K_q$ can be fully served by the depots in Q^{EC} and all clients in $\cup_{p\in P} K_p^{'}$ can be fully served by the depots in P. The excess capacity of the depots in P can be used to serve any client in $K_r^{'}$. Therefore,

$$f(J\backslash\{r\}) = \min(d(K), \sum_{l\in J\backslash\{r\}} \overline{m}_l) = \min(d(K), d(K) + \lambda - \overline{m}_r) = d(K) - (\overline{m}_r - \lambda)^+,$$

giving $\rho(J\backslash\{r\}) = (\overline{m}_r - \lambda)^+$.

Case 3: $r \in J^{EC}$ and $\{J^{EC}\backslash Q^{EC}\} \neq \{r\}$.

For the depots in $J^{EC}\backslash Q^{EC}$ we have $K_j = K^{EC}$. At least one depot in $J^{EC}\backslash Q^{EC}$ is open. The depots in Q^{EC} and P can fully serve the clients in $\cup_{q\in Q^{EC}} K_q$ and $\cup_{p\in P} K_p^{'}$ respectively. Hence, the excess capacity of the depots in P and the capacity of the depots in $J^{EC}\backslash Q^{EC}$ can be used to satisfy the demand of any client in $K^{EC}\backslash \cup_{q\in Q^{EC}} K_q$. Therefore,

$$f(J\backslash\{r\}) = \min(d(K), \sum_{l\in J\backslash\{r\}} \overline{m}_l) = d(K) - (\overline{m}_r - \lambda)^+,$$

giving $\rho(J\backslash\{r\}) = (\overline{m}_r - \lambda)^+$.

■

THEOREM 17. *Assume that* $f(J) - d(K) < \sum_{j\in J}\overline{m}_j$ *and that* $\sum_{j\in M} m_j > d(N) + m_r$ *for all* $r \in J$. *Then the submodular inequality*

$$\sum_{j\in J}\sum_{k\in K_j} v_{jk} + \sum_{j\in J}\rho_j(J\backslash\{j\})(1-y_j) \le f(J) \tag{4.3}$$

based on the single-depot structure, as defined in Lemma 16, defines a facet of $conv(X^{CFL})$ *if and only if*

a) $\rho_q(J\backslash\{q\}) > 0$ *for all* $q \in Q^{EC}$,

b) *if* $|Q^{EC}| \le 1$, *then* $\exists\ j \in J^{EC}\backslash Q^{EC}$ *with* $\rho_j(J\backslash\{j\}) > 0$.

Proof.

The proof essentially follows the same steps as the proof of Theorem 8. In addition to the tight points used in proving the sufficient conditions of Theorem 8, the following type of tight point is needed here.

(iv) **One depot** $p_1 \in P$ **is closed.**

$$\begin{aligned}
\sum_{j\in J\backslash\{p_1\}}\sum_{k\in K_j} v_{jk} &= f(J) - \rho_{p_1}(J\backslash\{p_1\}) = d(K) - d(K'_{p_1}) = d(\cup_{j\in J\backslash\{p_1\}}K_j) \\
\sum_{j\in M\backslash\{p_1\}}\sum_{k\in K'_{p_1}} v_{jk} &= d(K'_{p_1}) \\
\sum_{j\in M\backslash\{p_1\}}\sum_{k\in N\backslash K} v_{jk} &= d(N\backslash K) \\
\sum_{k\in N} v_{jk} &\le m_j - \varepsilon && j \in M\backslash\{p_1\} \\
y_j &= 1 && j \in M\backslash\{p_1\} \\
y_{p_1} &= 0 \\
v_{jk} &\ge \varepsilon && j \in M\backslash\{p_1\}, k \in K'_{p_1}.
\end{aligned}$$

■

Conditions *a)* and *b)* of Theorem 17 say that we should not add more depots to the set P than that the coefficients of y_j for $j \in Q^{EC}$ stay positive (cf. Conditions *d)* and *e)* of Theorem 8). If $P = \emptyset$, only the EC-component remains, and Conditions *a)* and *b)* are automatically satisfied.

The second structure having $\rho_j(J\backslash\{j\}) > (\overline{m}_j - \lambda)^+$ for at least one $j \in J$, is called the multi-depot structure and consists of a basic facet-defining EC-component and additional facet-defining EC-components C^i, $i = 1, ..., c$, with depot set J^i and client set K^i, all connected to the basic component.

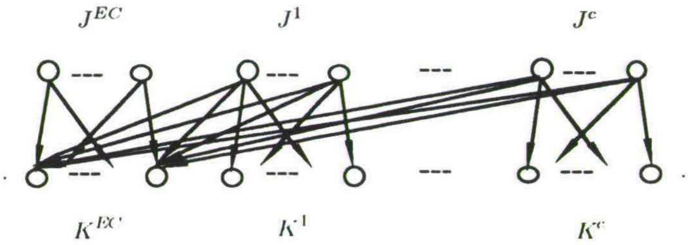

Figure 7.

LEMMA 18. *Let C^{EC} be a facet-defining EC-component, i.e. a component with client set K^{EC}, depot set J^{EC} and arc set $\{(j,k) : j \in J^{EC},\ k \in K_j \subseteq K^{EC}\}$ satisfying the conditions of Theorem 8. The set $Q^{EC} \subset J^{EC}$ is the set of depots in J^{EC} having $\overline{m}_j < m_j$. Let $\lambda^{EC} = \sum_{j\in J^{EC}} \overline{m}_j - d(K^{EC})$. Let C^i, $i = 1, ..., c$ be additional facet-defining EC-components with depot set J^i, client set K^i, arc set $\{(j,k) : j \in J^i,\ k \in K^i\}$ and excess $\lambda^i = \sum_{j\in J^i} \overline{m}_j - d(K^i)$. For each $j \in J^i$, $i = 1, ..., c$ redefine K_j to become $K_j := K^{EC} \cup K^i$. Let $J = J^{EC} \bigcup(\cup_{i=1}^{c} J^i)$, $K = K^{EC} \bigcup(\cup_{i=1}^{c} K^i)$ and $\lambda = \lambda^{EC} + \sum_{i=1}^{c} \lambda^i$.*

Then $\rho_j(J\backslash\{j\}) = (\overline{m}_j - \lambda)^+$ for all $j \in J^{EC}$ and $\rho_j(J\backslash\{j\}) = (\overline{m}_j - \lambda^i)^+$ for all $j \in J^i$, $i = 1, ..., c$. Moreover, if $\rho_j(J\backslash\{j\}) > 0$ for $j \in J^i$, then $\rho_j(J\backslash\{j\}) > (\overline{m}_j - \lambda)^+$.

Proof.

Close depot r.

Case 1: $r \in J^i$.

Due to the assumptions, all clients in $K\backslash K^i$ can be fully served by the depots in $J\backslash J^i$. Moreover, $\max(\sum_{j\in J^i\backslash\{r\}} \sum_{k\in K_j} v_{jk}) = \min(\sum_{j\in J^i\backslash\{r\}} \overline{m}_j, d(K^i))$ since C^i is a facet-defining EC component. Thus,
$f(J\backslash\{r\}) = d(K\backslash K^i) + \min(\sum_{j\in J^i\backslash\{r\}} \overline{m}_j, d(K^i)) = d(K\backslash K^i) + d(K^i) - (\overline{m}_r - \lambda^i)^+$,
which gives

$$\rho_r(J\backslash\{r\}) = f(J) - d(K) + (\overline{m}_r - \lambda^i)^+ = (\overline{m}_r - \lambda^i)^+.$$

$\lambda > \lambda^i$ and hence $(\overline{m}_r - \lambda^i)^+ > (\overline{m}_r - \lambda)^+$ if $\overline{m}_r > \lambda^i$.

Case 2: $r \in J^{EC}$ and $\{J^{EC} \backslash Q^{EC}\} = \{r\}$. Similar to the proof of Case 2, Lemma 16.

Case 3: $r \in J^{EC}$ and $\{J^{EC} \backslash Q^{EC}\} \neq \{r\}$. Similar to the proof of Case 3, Lemma 16.

■

THEOREM 19. *Assume that* $f(J) - d(K) < \sum_{j \in J} \overline{m}_j$ *and that* $\sum_{j \in M} m_j > d(N) + m_r$ *for all* $r \in J$. *Then the submodular inequality*

$$\sum_{j \in J} \sum_{k \in K_j} v_{jk} + \sum_{j \in J} \rho_j(J \backslash \{j\})(1 - y_j) \leq f(J) \tag{4.3}$$

based on the multi-depot structure, as defined in Lemma 18, defines a facet of $conv(X^{CFL})$ *if and only if*

a) $\rho_j(J \backslash \{j\}) > 0$ *for all* $j \in Q^{EC}$,

b) *if* $|Q^{EC}| \leq 1$, *then* $\exists\ j \in J^{EC} \backslash Q^{EC}$ *with* $\rho_j(J \backslash \{j\}) > 0$.

Proof.

The proof is similar to the proof of Theorem 17.

5. Combinatorial and Lot Sizing Inequalities

Here we discuss two families of valid inequalities; the class of combinatorial inequalities developed by Cho et al. (1983 a) for the uncapacitated facility location problem and the class of (k, l, S, I)-inequalities developed by Pochet and Wolsey (1993) for lot-sizing problems with constant batch sizes. Both families contain inequalities that are facet-defining for $conv(X^{CFL})$.

Let $K \subseteq N$ and define for each $j \in M$ a set $K_j \subseteq K$. Let $J \subseteq M$ be a set of depots such that each client in K is covered by at least one depot in J, i.e. $\cup_{j \in J} K_j = K$. Associated with the subgraph $G^S = G(V, E)$ where $V = \{j \in J\} \cup \{k \in K\}, E = \{(j,k) : j \in J,\ k \in K_j\}$ is an *adjacency matrix* $S = \{s_{jk}\}_{j \in J, k \in K}$ where

$$s_{jk} = \begin{cases} 1, & \text{if } k \in K_j \\ 0, & \text{otherwise.} \end{cases}$$

We assume that all rows of S are distinct.

Let $\beta(G^S)$ denote the *covering number* of G^S, i.e. the minimum number of depots in J necessary to cover all clients in K. Following the notation of Cho et al., an adjacency matrix S is called a *pd-adjacency matrix* if *i)* the corresponding subgraph G^S is connected, *ii)* there exists at least one zero element in each column, i.e. no client is connected to all depots in J, and *iii)* $|J| \geq 3$ and $|K| \geq 3$. Moreover, a *pd*-adjacency matrix is *maximal* if changing any zero element of S to one would decrease $\beta(G^S)$ by one. Let $k_{\beta-1}$ be any client in K and let $J^{\beta-1} \subset J$ be a subset such that $|J^{\beta-1}| = \beta - 1$ and such that the depots in $J^{\beta-1}$ cover all clients in K except client $k_{\beta-1}$. Similarly, let $J^{\beta} \subseteq J$ be a subset such that $|J^{\beta}| = \beta$ and such that the depots in J^{β} cover all clients in K. A consequence of the properties of a maximal *pd*-adjacency matrix (see Cho et al. (1983 b), Lemma 3.1) is that for each $k \in K$, there exists a set $J^{\beta-1}$ such that all clients except k are covered by the depots in $J^{\beta-1}$. Moreover, there exist subsets $J^{\beta-1}$ such that each depot for which $K_j \not\ni k$ belongs to at least one of the subsets $J^{\beta-1}$ and each depot $j \in J$ belongs to at least one set J^{β}.

Cho et al. (1983 b) showed that the combinatorial inequality

$$\sum_{j \in J} \sum_{k \in K_j} \frac{1}{d_k} v_{jk} - \sum_{j \in J} y_j \leq |K| - \beta(G^S) \tag{5.1}$$

defines a facet of the convex hull of feasible solutions to the uncapacitated location problem if and only if S is a maximal *pd*-adjacency matrix. Since the uncapacitated location problem is a relaxation of X^{CFL}, inequalities (5.1) are aso valid for $conv(X^{CFL})$. For $conv(X^{CFL})$ the following holds.

THEOREM 20. *The combinatorial inequality*

$$\sum_{j \in J} \sum_{k \in K_j} \frac{1}{d_k} v_{jk} - \sum_{j \in J} y_j \leq |K| - \beta(G^S) \tag{5.1}$$

defines a facet of $conv(X^{CFL})$ *if*

a) S *is a maximal pd-adjacency matrix,*

b) $m_j > d(K_j)$ *for all* $j \in J$,

c) $\sum_{j \in J^{\beta-1}} m_j + \sum_{j \in M \setminus J} m_j > d(N)$ *for all sets* $J^{\beta-1}$,

d) $\sum_{j \in J^{\beta}} m_j + \sum_{j \in M \setminus J} m_j - m_r \geq d(N)$ *for all sets* J^{β} *and all* $r \in M \setminus J$.

Proof. See Aardal (1992), Theorem 3.27.

In the case of constant capacities, $m_j = m$ for all $j \in M$, the following class of inequalities are adapted directly from the (k, l, S, I)-inequalities developed by Pochet and Wolsey (1993) for the lot-sizing problem with constant batch sizes.

Let $J \subseteq M$, $J = \{1, ..., |J|\}$. For each $j \in J$ define a set K_j such that $K_j \supset K_{j+1}$, $j = 1, ..., |J| - 1$. Figure 8 shows the structure of such sets J, K_j.

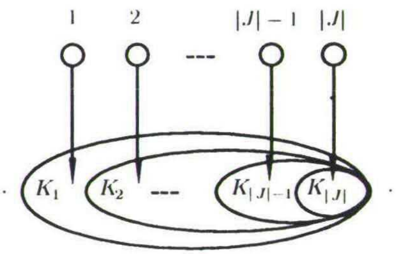

Figure 8.

Define $\eta_j = \lceil \frac{d(K_1 \setminus K_{j+1})}{m} \rceil$, $j = 1, ..., |J|-1$, $\eta_{|J|} = \lceil \frac{d(K_1)}{m} \rceil$ and $\gamma_j = d(K_1 \setminus K_{j+1}) - (\eta_j - 1)m$, $j = 1, ..., |J| - 1$, $\gamma_{|J|} = d(K_1) - (\eta_{|J|} - 1)m$. Let $I \subseteq J$ and define a permutation π on I such that $I = \{\pi_1, \pi_2, ..., \pi_{|I|}\}$ with $\gamma_{\pi_1} \leq \gamma_{\pi_2} \leq ... \leq \gamma_{\pi_{|I|}}$. By convention $\gamma_{\pi_0} = 0$. Let X_C^{CFL} denote the set of feasible solutions to CFL with $m_j = m$ for all $j \in M$ and let $Y(S) = \sum_{i \in S} y_i$ for any $S \subseteq J$.

PROPOSITION 21. *The inequality*

$$\sum_{j \in J} \sum_{k \in K_j} v_{jk} + \sum_{t=1}^{|I|} (\gamma_{\pi_t} - \gamma_{\pi_{t-1}})(\eta_{\pi_t} - Y(\{1, ..., \pi_t\})) \leq d(K_1) \tag{5.2}$$

is valid for $conv(X_C^{CFL})$.

EXAMPLE 2. Consider the following instance of the capacitated facility location problem with equal capacities.

Let $M = \{1,2,3,4\}$, $N = \{1,2,3,4\}$, $m = 5$, $d_1 = 1$, $d_2 = 2$, $d_3 = 4$, and $d_4 = 1$.

Choosing $J = \{1,2,3\}$, $K_1 = \{1,2,3\}$, $K_2 = \{2,3\}$, and $K_3 = \{3\}$ gives $\eta_1 = 1$, $\eta_2 = 1$, $\eta_3 = 2$, $\gamma_1 = 1$, $\gamma_2 = 3$, $\gamma_3 = 2$.

Let $I = J$ giving $\pi(1) = 1$, $\pi(2) = 3, \pi(3) = 2$ and the inequality

$$v_{11} + v_{12} + v_{13} + v_{22} + v_{23} + v_{33} + (1 - y_1) + (2 - y_1 - y_2 - y_3) + (1 - y_1 - y_2) \leq 7.$$

This inequality defines a facet of the convex hull of feasible solutions.

■

Inequalities (5.2), flow cover, EC and submodular inequalities are all of the form

$$\sum_{j\in J}\sum_{k\in K_j} v_{jk} \leq d(K) - \sum_t \beta_t(\eta_t - \sum_{j\in S_t} y_j).$$

They show that $\sum_{j\in J}\sum_{k\in K_j} v_{jk}$ is bounded by $d(K)$ if $\sum_{j\in S_t} y_j \geq \eta_t$ for all t, and also provide an upper bound on the flow $\sum_{j\in J}\sum_{k\in K_j} v_{jk}$ when one or several of the inequalities $\sum_{j\in S_t} y_j \geq \eta_t$ are violated. The significant difference between inequality (5.2) and the earlier inequalities is that both η_t and $|S_t|$ can be larger than one.

6. Extensions

In a companion paper we develop separation heuristics for the new families of strong valid inequalities and incorporate them in cutting plane algorithms to solve medium size problems.

Two possible extensions are first to find submodular structures other than the single and multi-depot structures, or a combination of them, that provides an explicit expression of $\rho_j(J\backslash\{j\})$ and that has $\rho_j(J\backslash\{j\}) > (\overline{m}_j - \lambda)^+$ for at least one $j \in J$. In this context it would also be interesting to investigate the relationship between the submodular inequalities and the (k, l, S, I)-inequalities. The structure of the (k, l, S, I)-inequalities resembles the submodular structures discussed earlier in that, on an aggregate level, there are arcs going in one direction only.

The other extension is to examine multi-level capacitated location problems and see how the inequalities discussed in this paper can be used to solve these more general problems as well as developing new inequalities for multi-level structures. Preliminary work in this direction can be found in Aardal (1992).

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References

Aardal, K. "On the solution of one and two-level capacitated facility location problems by the cutting plane approach," Ph.D. Thesis, Université Catholique de Louvain (Louvain-la-Neuve, 1992).

Cho, D.C, E.L. Johnson, M.W. Padberg and M.R. Rao, "On the uncapacitated plant location problem. I: Valid inequalities and facets," *Mathematics of Operations Research* 8 (1983 a) 579-589.

Cho, D.C., M.W. Padberg and M.R. Rao, "On the uncapacitated plant location problem. II: Facets and lifting theorems," *Mathematics of Operations Research* 8 (1983 b) 590-612.

Cornuéjols, G. and J.-M. Thizy, "Some facets of the simple plant location problem," *Mathematical Programming* 23 (1982) 50-74.

Cornuéjols, G., M. L. Fisher and G.L. Nemhauser, "On the uncapacitated location problem," *Annals of Discrete Mathematics* 1 (1977) 163-177.

Cornuéjols, G., R. Sridharan and J.M. Thizy, "A comparison of heuristics and relaxations for the capacitated plant location problem," *European Journal of Operational Research* 50 (1991) 280-297.

Crowder, H., E.L. Johnson and M.W. Padberg, "Solving large-scale zero-one linear programming problems," *Operations Research* 5 (1983) 803-834.

Deng, Q. and D. Simchi-Levi, "Valid inequalities, facets and computational results for the capacitated concentrator location problem," Research Report, Department of Industrial Engineering and Operations Research, Columbia University (New York, 1993).

Guignard, M., "Fractional vertices, cuts and facets for the simple plant location problem," *Mathematical Programming* 12 (1980) 150-162.

Leung, J.M.Y. and T.L. Magnanti, "Valid inequalities and facets of the capacitated plant location problem," *Mathematical Programming* 44 (1989) 271-291.

Nemhauser, G.L. and L.A. Wolsey, *Integer and Combinatorial Optimization* (John Wiley & Sons, Inc., New York, NY, 1988).

Padberg, M.W., T.J. Van Roy and L.A. Wolsey, "Valid inequalities for fixed charge problems," *Operations Research* 33 (1985) 842-861.

Pochet, Y. and L. A. Wolsey, "Lot-sizing with constant batches: Formulation and valid inequalities," *Mathematics of Operations Research* 18 (1993) 767-785.

Savelsbergh, M.W.P., G.C. Sigismondi and G.L. Nemhauser, "Functional description of MINTO, a Mixed INTeger Optimizer," Memorandum COSOR 91-17, Eindhoven University of Technology (Eindhoven, 1991).

Van Roy, T.J. and L. A. Wolsey, "Solving mixed integer programming problems using automatic reformulation," *Operations Research* 35 (1987) 45-57.

Welsh, D.J.A. *Matroid Theory* (Academic Press, 1976).

Wolsey, L.A. "Submodularity and valid inequalities in capacitated fixed charge networks," *Operations Research Letters* 8 (1989) 119-124.

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